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LETTER TO THE EDITOR

The cyclic boson operators and new representations of the quantum algebra $sl_q(2)$ for q a root of unity

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Abstract. The concept of the cyclic boson operator is introduced for the explicit construction of new representations of the quantum algebra $sl_q(2)$ when q is a root of unity. The method used here can easily be generalized to other quantum algebras or quantum superalgebras.

The study of the quantum group and its representation theory associated with the Yang–Baxter equation (YBE) is a hot subject [1–4], especially for the case that q is a root of unity (or $q^p=1$, $p=2, 3, \ldots$) [5–10]. More recently, De Concini and Kac systematically studied the irreducible representations of quantum universal enveloping algebras (quantum algebras) from a purely algebraic point of view when q is a root of unity [11]. Motivated by the problems in the Potts model, Date *et al* explicitly constructed such kinds of representations for $U_q(sl(n+1))$ and $U_q(A_2^{(2)})$ [12, 13]. Because the elements $(f)^p$ are non-zero constants in these representations where the generators f do not belong to the Cartan subalgebra, these representations are called cyclic representations.

In this letter we report how the q-deformed boson realization method constructing the representations of quantum algebras in generic cases ($q^p \neq 1$) [14–18] can be generalized to obtain such cyclic representations. We also obtain the cyclic representations with the multiplicities larger than one for quantum algebra $sl_q(2)$ as a completely new result, so far as we know. The discussion is followed with $sl_q(2)$ as example, but the method used can be applied to other quantum algebras or superalgebras.

The key to the study in this letter is the introduction of the cyclic boson operators. For the usual q-boson operators [14–16] $a^+=a^\dagger$, $a^-=a$ and N satisfying

$$aa^+ - q^{-1}a^+a = Q^+ \qquad [N, a^\pm] = \pm a^\pm \tag{1}$$

where we defined $Q^\pm = q^{\pm N}$. When $q^p=1$, we easily prove that $(a^\pm)^p$ and $(Q^\pm)^p$ commute with $a^\pm$ and $Q^\pm$; that is to say, they belong to the centre of the q-boson algebra [18] B_q generated by $a^\pm$ and $Q^\pm$. Thus, we can impose the additional relations

$$(a^\pm)^p = \mu_\pm \qquad \mu_+ = \mu \in \mathbb{C} \qquad \mu = \eta \in \mathbb{C} \qquad (Q^\pm)^p = 1 \tag{2}$$

($\mathbb{C}$ is the complex number field) on the algebra B_q when $q^p=1$ and there does not appear any contradiction. The operators $a^\pm$ and $Q^\pm$ satisfying (1) and (2) simultaneously are called cyclic boson operators while the finite-dimensional algebra B_c

with the basis

$$\left\{f(m,n,K)=\begin{cases}a^{+m}a^{n}(Q^{+})^{K}, K\geqslant 0\\ a^{+m}a^{n}(Q^{-})^{-K}, K<0\end{cases}\Bigg|, m, n, \pm K=0,1,2,\ldots,p-1\right\}$$

is called the cyclic boson algebra, which is generated by the cyclic boson operators.

Associated with the cyclic boson algebra B_c defined above, the cyclic Fock space F_c is defined as a span

$$\{F(m)=a^{+m}|0\rangle\,|\,a|0\rangle=0, Q^{\pm}|0\rangle=|0\rangle, m=0,1,2,\ldots,p-1\}$$

for $\eta=0$. This space naturely carries a p-dimensional representation:

$$\begin{aligned}&a^{+}F(m)=F(m+1) && 0\leqslant m\leqslant p-2\\ &a^{+}F(p-1)=\mu F(0)\\ &aF(m)=[m]F(m-1) && [m]\equiv(q^{m}-q^{-m})/(q-q^{-1})\\ &Q^{\pm}F(m)=q^{\pm m}F(m).\end{aligned}\tag{3}$$

Now, we consider the quantum algebra $sl_q(2)$ generated by $J_\pm$ and J_0 (or $K^\pm=q^{\pm 2J_0}$), and the relations

$$\begin{aligned}&[J_+,J_-]=[2J_0]\ (=(K^+-K^-)/(q-q^{-1}))\\ &[J_0,J_\pm]=\pm J_\pm \qquad (\text{or } KJ_\pm=q^{\pm 2}J_\pm K, K^+=K, K^-=K^{-1}).\end{aligned}\tag{4}$$

It follows from (1) and (3) that, on the space F_c,

$$J_+=a^+ \qquad J_-=\frac{a(q^{\lambda+1}Q^- - q^{-\lambda-1}Q^+)}{q-q^{-1}} \qquad K^\pm=(Q^\pm)^2q^{\mp\lambda}\tag{5}$$

satisfy the basic relations (4); namely, equations (5) define an operator representation of $sl_q(2)$, which is called the cyclic boson realization of $sl_q(2)$. From this realization, we immediately get an irreducible representation of $sl_q(2)$ on F_c through the representation (3) of B_c as follows

$$\begin{aligned}&J_+F(m)=F(m+1) && 0\leqslant m\leqslant p-2\\ &J_+F(p-1)=\mu F(0)\\ &J_-F(m)=[m][\lambda+1-m]F(m-1)\\ &K^\pm F(m)=q^{\pm 2m\mp\lambda}F(m).\end{aligned}\tag{6}$$

Then, we reconstruct the cyclic representation first given by De Concini and Kac (see remark 4.2(b) in [11]) in a special case that one of two parameters in their representation is taken to be zero. In an analogue

$$\Psi_j(m)=\left\{\prod_{K=-j+1}^{m}([j+K][\lambda-j-K+1])\right\}^{-1/2}F(j+m)$$

$$\Psi_j(-j)=F(0) \qquad m=-j+1,-j+1,-j+2,\ldots,j \qquad \text{for } j=(p-1)/2$$

of the angular momentum basis for SO(3), the representation (6) is rewritten as follows:

$$\begin{aligned}&J_+\Psi_j(m)=\{(\lambda-j-m][j+m+1]\}^{1/2}\Psi_j(m+1) && -j\leqslant m\leqslant j-1\\ &J_-\Psi_j(m)=\{[j+m][\lambda-j-m+1]\}^{1/2}\Psi_j(m-1)\\ &K^\pm\Psi_j(m)=q^{\pm 2(j+m)\mp\lambda}\Psi_j(m)\\ &J_+\Psi_j(j)=\tilde{\mu}\Psi_j(-j)\\ &\tilde{\mu}=\left\{\prod_{K=-j+1}^{j}([j+k][(\lambda-j)-K+1])^{-1/2}\right\}\cdot\mu.\end{aligned}\tag{7}$$

Now, let us try to construct a new cyclic representation of $sl_q(2)$, which has not been given in an explicit form by any other author up to now. To this end we first give another cyclic boson realization

$$\begin{aligned} &J_+ = a_1^+ \qquad K^{\pm} = (Q_1^{\pm})^2 (Q_2^{\mp})^2 q^{\pm\lambda} \\ &J_- = a_2^+ + a_1[2N_2 - N_1 + 1 + \lambda] \qquad \lambda \in \mathbb{C} \end{aligned} \tag{8}$$

on the two-state cyclic Fock space $F_c(2)$:

$$\{F(m, n) = a_1^{+m} a_2^{+n}|0\rangle \mid a_i|0\rangle = 0, Q_i^{\pm}|0\rangle, i = 1, 2; m, n = 0, 1, 2, \ldots\}$$

corresponding to the two-state cyclic boson algebra $\mathbf{B}_c(2)$ with the generator $a_i = a_i^-$, a_i^+ and $Q_i^{\pm}$ $(i = 1, 2)$. Here, we follow the original definitions about the multi-state case in three early works [14-16] and let the boson operators of different states commute with each other, i.e.

$$\begin{aligned} &a_i a_i^+ - q^{-1} a_i^+ a_i = Q_i = q^{N_i} \\ &[N_i, a_i^{\pm}] = \pm a_i^{\pm} \qquad i = 1, 2 \\ &[x_j, y_i] = 0 \qquad i \neq j \qquad x, y = a^{\pm}, Q^{\pm} \end{aligned}$$

where the cyclic conditions are $(a_i^+)^p = \mu \in \mathbb{C}$ $(i = 1, 2)$. Then, on $F_c(2)$, we obtain a p^2-dimensional representation

$$\begin{aligned} &J_+ F(m, n) = F(m+1, n) \qquad 0 \leqslant m \leqslant p-2 \\ &J_+ F(p-1, n) = \mu_1 F(0, n) \\ &J_- F(m, n) = F(m, n+1) + [m][2n - m + 1 + \lambda] F(m-1, n) \qquad 0 \leqslant n \leqslant p-2 \\ &J_- F(m, p-1) = \mu_2 F(m, 0) + [m][\lambda - m - 1] F(m-1, p-1) \\ &K^{\pm} F(m, n) = q^{\pm 2(m-n) \pm \lambda} F(m, n). \end{aligned} \tag{9}$$

It is irreducible if $\mu_i \neq 0$ and λ is not an integer. Because there exist the weights with multiplicities larger than one, it cannot be covered by the work of De Concini and Kac [11] in an explicit form. For example, the weight vectors $F(m+k, m)$ $(m = 0, 1, 2, \ldots, p-1)$ correspond to the same weight $2k - \lambda$. In fact, the formal cyclic conditions

$$F(p, n) = \mu_1 F(0, n) \qquad F(m, p) = \mu_2 F(m, 0)$$

enable one to represent the basis $F(m, n)$ by the lattice point (m, n) on a torus $S^1 \times S^1 : \{(m, n) \mid (m+p, n+p) = (m, n)\}$, but the representation given in [11] only corresponds to a circle. Therefore, we say the two representations, given by us and by De Concini and Kac, have different 'topologies'.

Finally, we study another representation

$$\begin{aligned} &J_+ F(m, n) = [n] F(m+1, n-1) \qquad 0 \leqslant m \leqslant p-2 \qquad 1 \leqslant n \leqslant p \\ &J_+ F(p-1, n) = [n] \mu_1 F(0, n-1) \qquad 1 \leqslant n \leqslant p \\ &J_- F(m, n) = [m] F(m-1, n+1) \qquad 0 \leqslant n \leqslant p-2 \\ &J_- F(m, p-1) = [m] \mu_2 F(m-1, 0) \\ &K^{\pm} F(m, n) = q^{\pm(m-n)} F(m, n) \end{aligned} \tag{10}$$

of $sl_q(2)$ on the space $F_c(2)$ in terms of another realization

$$J_+ = a_1^+ a_2 \qquad J_- = a_2^+ a_1 \qquad K^{\pm} = Q_1^{\pm} Q_2^{\mp}. \tag{11}$$

In order to reduce this representation, we formally extend $F_c(2)$ to the space $\tilde{F}_c(2)$: $\{F(m, n) \in F_c(2), F(p, n), F(m, p)\}$. Limited by the cyclic conditions $F(m, p) = \mu_2 F(m, 0)$ and $F(p, n) = \mu_1 F(0, n)$, $\tilde{F}_c(2)$ returns to $F_c(2)$. Then, we immediately find an invariant subspace V: $\{F(m, n) \in F_c(2) | m+n = N \text{ or } N+p\}$ for $N < p$. Here, we have considered the connections $F(N, p) = \mu_2 F(N, 0)$ and $F(p, N) = \mu_1 F_{10}, N)$. If $F(m, n)$ are also denoted by a lattice point (m, n) on the above defined torus $S^1 \times S^2$, then the invariant subspace V is denoted by a set of lattice points

$$(0, N) \to (1, N-1) \to \cdots \to (n, 0) \sim (N, p) \to (N+1, p-1) \to \cdots$$
$$\to (p-1, N+1) \to (P, N) \sim (0, N)$$

on a closed line on the torus $S^1 \times S^1$. Considering the dimension p of V is independent of N, we choose p vectors

$$f_N(m) = \begin{cases} F(m, N-m) & 0 \leqslant m \leqslant N \\ F(m-1, N+P+1-m) & N+1 \leqslant m \leqslant p+1 \end{cases}$$

as the basis for invariant subspace V where $f_N(N+1) = \mu_2 f_N(N), f_N(p+1) = \mu_1 f_N(0)$. Then, we explicitly write down the p-dimensional representation on it

$$\begin{aligned}
&J_+ f_N(m) = [N-m] f_N(m+1) \qquad 0 \leqslant m \leqslant N-1 \qquad m \pm p \\
&J_+ f_N(N) = 0 \\
&J_+ f_N(p) = [N+1] \mu_1 f_N(0) \\
&J_- f_N(m) = [m] f_N(m-1) \qquad 1 \leqslant m \leqslant N+1 \\
&J_- f_N(0) = 0 \\
&J_- f_N(N+2) = [N+1] \mu_1 f_N(N) \\
&J_+ f_N(m) = [N+1-m] f_N(m+1) \qquad N+1 \leqslant m \leqslant p-1 \\
&J_- f_N(m) = [m-1] f_N(m-1) \qquad N+3 \leqslant m \leqslant p \\
&K^\pm f_N(m) = q^{\pm 2m \mp N} f_N(m) \\
&K^\pm f_N(m) = q^{\pm 2m \mp (N+2)} f_N(m).
\end{aligned} \tag{12}$$

Because the complementary space $\{f_N(N+2), f_N(N+3), \ldots, f_N(p)\}$ and any others to the invariant subspace $\{f_N(0), f_N(1), \ldots, f_N(N)\}$ are not invariant, the representation (12) is indecomposable (reducible, but not completely reducible). The similar circumstance can appear for the general representations in remark 4.2 of [11] when the special parameters are taken.

Based on this letter, the systematic works on the q-boson realization of cyclic representations for quantum algebras and superalgebras are being prepared for publication. The new representations obtained in this letter can be used for the construction of new solutions of the YBE through the scheme in [13]. Finally, it is pointed out that the main results and the central idea are not covered by some recent studies [19–20] by other authors.

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